# CONFORMAL AND BROWNIAN MARTINGALES, INTEGRAL REPRESENTATION

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## 1. Introduction and some background

Conformal martingales are two dimensional stochastic processes that arise as a composition of an analytic/ holomorphic function and a complex Brownian motion. They are being studied since at least the 70s (as far as I know). They enjoy some interesting and curious properties that we shall investigate and they are useful in diverse settings like artificial intelligence [Vov19] and the study of harmonic functions on Riemann manifolds [Le99].

Since we are going to be using the complex representation for two dimensional stochastic processes and encounter some concepts in complex analysis we need to recall some basics. A holomorphic function on an open neighbourhood of $\mathbb{C}$ is a function $f = p + iq$ which is continuously differentiable as a function of two real variables and such that the derivative is a complex multiplication (Cauchy-Riemann conditions)

$$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y} \quad \text{and} \quad \frac{\partial q}{\partial x} = -\frac{\partial p}{\partial y}$$

such a function is called $\mathbb{C}$-differentiable and we have $f' = \frac{\partial p}{\partial x} - i\frac{\partial p}{\partial y}$. We define the following differential operators

$$\frac{\partial}{\partial z} := \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} := \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$

The Cauchy Riemann conditions translate to $\frac{\partial f}{\partial \overline{z}} = 0$ and we have $f'(z) = \frac{\partial f}{\partial z}$. We recall also that the zeroes of a holomorphic function are isolated (no accumulation points). A function that is holomorphic on the whole entire plane is called an **entire** function.

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We will be revisiting some concepts from a previous talk on time changes for continuous semi-martingales hence we recall some important theorems that we have seen

**Theorem 1.1.** *(Dambis, Dubins-Schwarz) Let $X_t$ be a local martingale vanishing at 0 with $\langle M, M\rangle_\infty = \infty$. Then if we set*

$$T_t = \inf\{s : \langle M, M\rangle_s > t\} \quad \textit{and } B_t = X_{T_t}$$

*Then $B_t := M_{T_t}$ is a Brownian motion and $M_t = B_{\langle M,M\rangle_t}$.*

**Theorem 1.2.** *(Knight) Let $X_t$ be a $d$-dimensional continuous local martingale vanishing at 0 with $\langle X_i, X_j\rangle = 0$ for $i \neq j$ and $\langle X_i, X_i\rangle_\infty = \infty$. Then if we set*

$$T^k_t = \inf\{s : \langle X^{(k)}, X^{(k)}\rangle_s > t\} \quad \textit{and } B^{(k)}_t = X^{(k)}_{T^k_t}$$

*Then $B$ is a Brownian motion*

## 2. Conformal martingales and planar Brownian motion

A two dimensional local martingale $Z$ is a two dimensional stochastic process whose coordinates are local martingales with respect to the same filtration. We use the complex representation and write $Z = X + iY$ where $X, Y$ are local martingales. For example the planar Brownian motion $B = B^{(1)} + iB^{(2)}$ is a two dimensional martingale. As we have seen before one can define the bracket of two real valued continuous semi-martingales using quadratic variation. The same can be done for a complex valued continuous local martingale.

**Proposition 2.1.** *Let $Z$ be a continuous local martingale. There exists a unique continuous complex complex process of finite variation which we denote by $\langle Z, Z\rangle$ such that $Z^2 - \langle Z, Z\rangle$ is a complex local martingale.*

*Proof.* Uniqueness is easy to see using the fact that a real-valued local-martingale for which the bracket is 0 is constant (this we have seen in a previous talk). For existence one just need to define the bracket by $\mathbb{C}$-linearity in both variables i.e

$$\langle Z, Z\rangle = \langle X + iY, X + iY\rangle := \langle X, X\rangle - \langle Y, Y\rangle + 2i\langle X, Y\rangle$$

It is clear that this process has finite variation and that $Z^2 - \langle Z, Z\rangle$ is a local martingale. □

Notice that for $Z^2$ to be a local martingale it is necessary and sufficient to have $\langle Z, Z\rangle = 0$ which is equivalent to $\langle X, X\rangle = \langle Y, Y\rangle$ and $\langle X, Y\rangle = 0$.

**Definition 2.2.** If one the previous equivalent conditions holds we call $Z$ a conformal local martingale (conf. loc. mart)

The planar Brownian motion is clearly a conf. loc. mart. and also is $H = H^X + iH^Y$ is a complex locally bounded predictable process and $Z$ a conf. loc. mart. the process

$$U_t := \int_0^t H_s dZ_s := \int_0^t H^X_s dX_s - \int_0^t H^Y_s dYs + i\left(\int_0^t H^X_s dY_s + \int_0^t H^Y_s dXs\right)$$

is a loc. conf. mart. This is not so hard to see using the fact that the $\mathbb{C}$-linearity of the bracket. Notice also that one has $\langle X, X\rangle = \frac{1}{2}\langle Z, \overline{Z}\rangle$ and hence we have in particular

$$\begin{aligned}\langle U, \overline{U}\rangle_t &= 2\Big\langle \int_0^\cdot H_s^X dX_s - \int_0^\cdot H_s^Y dYs, \int_0^\cdot H_s^X dX_s - \int_0^\cdot H_s^Y dYs\Big\rangle_t \\ &= 2\int_0^t (H_s^X)^2 d\langle X, X\rangle_s + \int_0^t (H_s^X)^2 d\langle Y, Y\rangle_s = \int_0^t |H_s|^2 d\langle Z, \overline{Z}\rangle_s\end{aligned}$$

We have a simple formulation of Ito's formula in the

**Proposition 2.3.** *If $Z$ is a conf. loc. mart and $F$ a complex valued function on $\mathbb{C}$ which is twice continuously differentiable then we have*

$$F(Z_t) = F(Z_0) + \int_0^t \frac{\partial F}{\partial z}(Z_s)dZ_s + \int_0^t \frac{\partial F}{\partial \overline{z}}(Z_s)d\overline{Z}_s + \frac{1}{4}\int_0^t \Delta F(Z_s)\langle Z, Z\rangle_s$$

*if $F$ is harmonic then we can see that $F$ is a local martingale. Moreover if $F$ is a holomorphic function we have*

$$F(Z_t) = F(Z_0) + \int_0^t F'(Z_s)dZ_s$$

We revisit Knight's theorem (seen in a previous talk on time changes) in the conformal setting

**Theorem 2.4.** *If $Z$ is a conformal local martingale and $Z_0 = 0$ then there exists (with possible enlargement of probability space) a complex Brownian motion $B$ such that*

$$Z_t = B_{\langle X, X\rangle_t}$$

*Proof.* Follows easily from DDS or Knight's theorem. □

One consequence of this theorem is what's called the *conformal invariance of planar Brownian motion* which states that

**Theorem 2.5.** *If $F$ is a non constant entire function, $F(B_t)$ is a time changed BM. More precisely there exists on the probability space of $B$ a complex Brownian motion $\tilde{B}$ such that*

$$F(B_t) = F(B_0) + \tilde{B}_{\langle X, X\rangle_t}$$

*where $\langle X, X\rangle_t = \int_0^t |F'(B_s)|^2 ds$ is strictly increasing and $\langle X, X\rangle_\infty = \infty$*

*Proof.* $F^2$ is also entire and hence $F^2(B_t)$ is a loc. mart. using Ito's formula. Hence $F \circ B$ is a conf. loc. mart. Hence by the previous theorem one has $F(B_t) = F(B_0) + \tilde{B}_{\langle X, X\rangle_t}$ where $X$ is the real part of $F(B)$ and thus

$$\langle X, X\rangle_t = \int_0^t |F'(B_s)|^2 ds$$

Since there is only countably many zeros of $F'$ then $B$ almost surely avoids all of them hence $\langle X, X\rangle_t$ is strictly increasing almost surely. Remains to show that it's limit is $\infty$. We will come back to this later on when we have the required information to deal with it. □

For a Markov process $X$ on a state space $E$ we call a Borel set $A$ polar if for any starting point $z$ of the process $X$ the hitting probability of $A$ is 0 i.e

$$\forall z \in E, \quad \mathbb{P}_z(T_A < \infty) = 0$$

We have seen that the Brownian motion is one dimension is point-recurrent . The following result show that the one-point sets are polar for $d \geq 2$

**Theorem 2.6.** *The one-point sets are polar for the Brownian motion in dimension $d \geq 2$*

*Proof.* It is enough to show it for $d = 2$ and using the symmetries and geometric properties of Brownian motion it's enough to show that the BM started at 0 does not hit the point $-1$ a.s.

Let $M_t = e^{B_t} - 1$, since the exponential is entire the previous theorem allows us to write $M_t = \tilde{B}_{A_t}$ where $\tilde{B}$ is a planar BM and $A_t = \int_0^1 \exp(2X_s)$ where $X = Re(B)$. The process $A$ is strictly increasing and $A_\infty = \infty$ since if not $M$ has an "end" point in $\mathbb{C}$ but $|M_t| \geq \exp(X_t) - 1$ and surely $X_t$ is not bounded and is recurrent which is then a contradiction. So the paths of $M$ are exactly the Brownian paths. But since $exp(B_t)$ is never 0 this shows that these paths avoid the point $-1$ a.s. □

This one among many proofs and the nice thing about it is that it is very simple and slick but that is due to the heavy machinery we already developed. This shows how important the time change idea is: it makes life easier.

So one-point sets are polar or Brownian motion of dimension $d \geq 2$ and hence so are all the countable sets (but there also uncountable set that are polar even in $d = 2$). Another interesting property of Brownian motion is that it is neighbourhood-recurrent in $d = 2$ (but not in $d \geq 3$ !) as the following theorem states:

**Theorem 2.7.** *Let $B$ be a planar BM. For any point $z$ and any $r > 0$ the set $\{t, B_t \in B(z, r)\}$ is unbounded.*

*Proof.* Consider $M_t = \log(|B_t - z|)$ this is a.s well defined since $\{z\}$ is polar. By the "same argument" as above we can write $M_t = W_{A_t}$ where $W$ is a BM in dimension 1 and $A_t = \int_0^t \frac{1}{|B_s - z|^2} ds$. Since $|B_t - z| \geq |X_t - x|$ which is not bounded hence $\sup M_t = +\infty$ which gives $A_\infty = \langle M, M \rangle_\infty = \infty$ and surely $A$ is strictly increasing hence the paths of $M$ are exactly the paths of a Brownian motion in dimension 1 which goes below $\log(r)$ at arbitrarily large $t$. □

**Remark 2.8.**
1) Other proofs of this exist for instance using harmonic functions and Ito's formula $\log(|z|)$ is harmonic in $d = 2$ hence its frequent appearance in $d = 2$.

2) A more powerful result exists: For any Borel set $A$ with strictly positive Lebesgue measure the set $\{t, B_t \in A\}$ is unbounded and even better of infinite Lebesgue measure.

3) The Brownian path on the plane is a.s dense!

Now that we have seen some interesting results we can go back to finish the proof of Theorem 2.5.

Suppose we had $\langle X, X\rangle_\infty < \infty$. Then the process $F(B_t)$ would have a limit at $\infty$. Hence $F(B_t)$ is as bounded on a positive measure set. Since $B_t$ is dense in $d = 2$ this can only mean that $F$ is bounded and entire. A theorem of Liouville concerning entire functions implies then that $F$ is constant which is a contradiction.

The following result deals with the transience of BM in $d \geq 3$ and the statement is as follows

**Theorem 2.9.** *If $d \geq 3$ then $\lim |B_t| = +\infty$ a.s.*

*Proof.* Clearly it is enough to show this in $d = 3$ when $B$ is started at some point $x_0 \neq 0$. Since $\{0\}$ is polar Ito's formula on $\frac{1}{|B_t|}$ shows that this is a positive loc. mart hence a positive super-martingale which thus converges as to a non-negative random variable $H$. Fatou's lemma shows that

$$\mathbb{E}_{x_0}[H] \leq \liminf E_{x_0}[\frac{1}{|B_t|}]$$

But $\frac{1}{|B_t|} \overset{d}{=} \frac{1}{|B_1|\sqrt{t}}$ so we deduce that the above liminf is 0 hence $H = 0$ a.s. Hence $|B_t|$ converges to $\infty$ a.s. $\square$

It might be unclear where the ingredient $d \geq 3$ is used in the previous argument. Actually it is hidden in the fact that $f_d(x) := \frac{1}{|x|^{d-2}}$ is harmonic for $d \geq 3$.

Here is a curious result on conformal martingales

**Theorem 2.10.** *Let $Z$ be a conformal martingale on $(0, +\infty)$ (i.e where we don't start anywhere : no starting point). Then for almost every $\omega$ on of the following happen*

(1) $\lim_{t\to 0^+} Z_t(\omega)$ *exists in the Riemann sphere ($\mathbb{C}$ together with its boundary).*
(2) *For each $\delta > 0$ the set $\{Z_t(\omega), 0 < t < \delta\}$ is dense in $\mathbb{C}$.*

**Remark 2.11.** Both possibilities can occur. If $B_t$ is a planar BM starting at 0 and $f$ is a holomorphic function on $\mathbb{C}^*$ then $f(B_t), t > 0$ is a conformal martingale. If 0 is a removable singularity ($f$ bounded near 0) then $\lim f(B_t)$ exists. If the singularity is a pole then $\lim f(B_t) = \infty$ but if it is an essential singularity then the set $\{f(B_t), 0 < t < \delta\}$ is dense for any $\delta > 0$.

A very interesting example is $f(z) = e^{1/z}$ which has an essential singularity in 0. The above then says that $\{e^{1/B_t}, 0 < t < \delta\}$ is dense in $\mathbb{C}$ for any $\delta > 0$.

*Proof.* Omitted. See [DMW77] page 491. $\square$

This is an analogue for conformal martingales of Weierstrass's theorem on essential singularities.

We now discuss the polar representation of the planar/complex Brownian motion started at some point $a \neq 0$. Since $\{0\}$ will then be polar one can choose a continuous determination $\theta$ of the argument of $B$ such that $\theta_0$ is a constant $\exp(i\theta_0) = \frac{a}{|a|}$. We then write $B_t = \rho_t \exp(i\theta_t)$ and the two processes $\rho, \theta$ are adapted to the filtration generated by $B$. Now here is a theorem

**Theorem 2.12.** *There exists a planar BM* $(\beta, \gamma)$ *such that*

$$\rho_t = |a| \exp(\beta_{C_t}) \text{ and } \theta_t = \theta_0 + \gamma_{C_t}$$

*where* $C_t := \int_0^t \rho_s^{-2} ds$. *We also have* $\mathcal{F}_\infty^\beta = \mathcal{F}_\infty^\rho$ *hence* $\rho$ *is independent of* $\gamma$.

*Proof.* Since $B$ starts from $a \neq 0$ it a.s never hits 0 so one can define a conformal martingale by

$$H_t := \int_0^t B_s^{-1} dB_s$$

Hence $\frac{1}{2}\langle H, \overline{H} \rangle = \langle Re(H), Re(H) \rangle = C$. We claim that $B = a \exp(H)$ this can be proved using Ito's formula for conf. loc. mart. via

$$d(B_t e^{-H_t}) = 0$$

There exists a planar BM $(\beta, \gamma)$ such that $H_t = \beta_{C_t} + i\gamma_{C_t}$. This shows half of the statement above. For the second statement notice that $\beta$ is the DDS Brownian motion for the loc. mart. $Re(H) = \log(\frac{\rho_t}{|a|})$ and

$$Re(H_t) = \int_0^t \frac{X_s dY_s + Y_s dX_s}{\rho_s^2} = \int_0^t \frac{d\tilde{\beta}_s}{\rho_s}$$

where $\tilde{\beta}$ is a real BM. We then get

$$\log(\rho_t) = \log(|a|) + \int_0^t \exp(-\log(\rho_s)) d\tilde{\beta}_s$$

Then from proposition $V.1.11$ in [RY13] we get $\mathcal{F}_\infty^\beta = \mathcal{F}_\infty^\rho$ and thus $\rho$ is independent of $\gamma$. □

Notice that as $\rho$ becomes small the argument $\theta$ varies more rapidly which is intuitive. Also we have

$$\liminf \theta = -\infty \text{ and } \limsup \theta = +\infty \text{ a.s}$$

this means that the planar BM winds an arbitrarily large numbers of times around 0 then unwinds an arbitrarily large number of times and keeps doing this forever. Some strong asymptotic results concerning $\theta$ have been established among which me mention

**Theorem 2.13.** *(Spitzer's law)* $\frac{2\theta_t}{\log(t)} \xrightarrow[t\to\infty]{d} C_1$ *where* $C_1$ *has the standard Cauchy distribution.*

*Proof.* See [PY18][section 8.3] and [Spi91] □

The limit as one can expect does not depends on the starting point $z_0 \neq 0$, but it's curious that the Cauchy distribution appears as a limit.

There is much much more to planar Brownian motion than what we have just seen (intersections, windings, asymptotic laws, ... etc). For more details on these we refer to the *Guide to Brownian motion* [PY18] and to [Spi91].

A cute application of this theory is the proof of the fundamental theorem of algebra (or D'Alembert's theorem) which states that $\mathbb{C}$ is an algebraically closed field (each non constant polynomial with complex coefficients has a root in $\mathbb{C}$). For any non constant polynomial $P$ the set paths of the process $P(B_t)$ are the Brownian paths in the plane and we know these are neighbourhood recurrent hence a.s the set $\{t > 0, |P(B_t)| \leq \epsilon\}$ is unbounded. This proves the existence of a sequence $z_k \in \mathbb{C}$ such that $|P(z_k)| \leq 1/k$ for all $k$. Since $P$ explodes at $\infty$ the points $z_k$ are all in some compact region of plane hence there is a sub-sequence that converges to a root of $P$.

## 3. Brownian martingales

Following our previous discussion of conformal martingales, in this section we discuss Brownian martingales and some first results on stochastic integral representation. First of all we set things up by recalling that we are working on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ on which we have an $(\mathcal{F}_t)$-Brownian motion $B$ and we assume for simplicity of notation that the filtration $(\mathcal{F})_t$ is generated by $B$ (suffices to change the filtration otherwise) and $\mathcal{F}_\infty = \mathcal{F}$.

We call $\mathcal{S}$ the set of left continuous step function with compact support by which we mean functions $f$ of the form

$$f = \sum_{i=1}^{n} \lambda_i 1_{]t_{i-1},t_i]}$$

and for any $f \in \mathcal{S}$ we denote by $\mathcal{E}^f$ the process defined as

$$\mathcal{E}^f_t := \exp\left(\int_0^t f(s)dBs\right)$$

The process $\mathcal{E}^f$ obviously has a limit when $t \to \infty$ which we simply denote by $\mathcal{E}^f_\infty$. Let $L^2(\mathcal{F}, \mathbb{P})$ be the space of square integrable functions on our probability space.

**Lemma 3.1.** *The set $\{\mathcal{E}^f_\infty, f \in \mathcal{S}\}$ is total in $L^2(\mathcal{F}, \mathbb{P})$. (i.e generates a dense subspace of)*

*Proof.* We proceed by showing that if $Y \in L^2(\mathcal{F}, \mathbb{P})$ is orthogonal to all the variables $\mathcal{E}^f_\infty$ then $Y$ is $\mathbb{P}$-a.s zero which means showing that $Y.P$ is the zero measure. This a standard argument when dealing with Hilbert spaces and we are going to rely on some notions of complex analysis. First we show that $Y.P$ is zero on $\sigma(B_{t_0}, B_{t_2} \dots, B_{t_n})$ for $t_0 < \dots < t_n$.

For $z_1, \dots, z_n \in \mathbb{C}$ let $\phi(z_1, \dots, z_n) := \mathbb{E}\left[\exp\left(\sum_{j=1}^n z_j(B_{t_j} - B_{t_{j-1}})\right) Y\right]$. Clearly $\phi$ is an analytic function on $\mathbb{C}^n$ (locally a power series in multiple variables at each point). Saying that $Y$ is orthogonal to all the $\mathcal{E}^f_\infty$'s implies that $\phi$ is identically zero on $\mathbb{R}^n$ and since it's analytic this implies that $\phi \equiv 0$ everywhere. In particular when $z_j = i\lambda_j$ for $\lambda_j \in \mathbb{R}$ we get

$$\mathbb{E}\left[\exp\left(\sum_{j=1}^n i\lambda_j(B_{t_j} - B_{t_{j-1}})\right) Y\right] = 0$$

Hence the Fourier transform of the measure $Y.P$ is identically 0 on $\sigma(B_{t_0}, B_{t_1} \dots, B_{t_n})$ which means that $Y.P = 0$ on $\sigma(B_{t_0}, B_{t_1} \dots, B_{t_n})$. This implies that $Y = 0$ a.s on $\mathcal{F}$. $\square$

This lemma will turn out to be very useful to show the

**Proposition 3.2.** *For any $Y \in L^2(\mathcal{F}, \mathbb{P})$ there exists a* ***unique*** *predictable process $H$ in $L^2(B)$ ($L^2$ processes with Brownian filtration) such that*

$$F = \mathbb{E}[F] + \int_0^\infty H_s dB_s$$

*Proof.* Let $\mathcal{H}$ be the subspace of $L^2(\mathcal{F}, \mathbb{P})$ consisting of elements that can be written as above (elements that are representable as stated). The strategy to prove existence is to show that this space contains the total set of the previous lemma and that it is closed. Let's deal with uniqueness first. For an $F \in \mathcal{H}$ we have

$$\mathbb{E}[F^2] = \mathbb{E}[F]^2 + \mathbb{E}\left[\int_0^{+\infty} H_s^2 ds\right]$$

Hence if the $F$ is representable by two process $H$ and $H'$ one would get (this this representation is linear)

$$\mathbb{E}\left[\int_0^{+\infty} |H_s - H_s'|^2 ds\right] = 0$$

Now as explained to show existence we start by noticing that $\mathcal{H}$ contains the $\mathcal{E}_\infty^f$'s for $f \in \mathcal{S}$ since we have using Ito's formula that

$$\mathcal{E}_t^f = 1 + \int_0^t \mathcal{E}_s^f f(s) dB_s$$

It remains then to show that $\mathcal{H}$ is a closed space (even better, we will show that $\mathcal{H}$ is complete). Let $(F_n)$ be a Cauchy sequence in $\mathcal{H}$ with corresponding processes $(H_n)$ which is Cauchy in $L^2(B)$. The latter space being complete the sequence $(H_n)$ converges to a predictable process $H \in L^2(B)$ hence $F_n$ converges to

$$\lim \mathbb{E}[F_n] + \int_0^{+\infty} H_s dB_s$$

This finishes the proof. □

An important thing to notice is that the condition $H \in L^2(B)$ is crucial for uniqueness. For instance let $T > 0$ and $d_T$ the first time $B$ hits 0 after $T$ i.e $d_T := \{u > T, B_u = 0\}$ then $F = 0$ can be represented by the zero process and by $H = 1_{[0,d_T]}$ (but $H$ is not in $L^2(B)$ in this case).

The main result of this section will be the extension of the previous proposition to local martingales which is the statement of the

**Theorem 3.3.** *Let $M$ be an $(\mathcal{F}_t)$-local martingale. Then $M$ has a version that can be written as*

$$M_t = C + \int_0^{+\infty} H_s dB_s$$

*where $C$ is a constant and $H$ is a predictable process which is locally in $L^2(B)$. In particular every $(\mathcal{F}_t)$-local martingale has a continuous version.*

*Proof.* The proof is divided into three steps. First we show the theorem for $L_2$ bounded martingales then for uniformly integrable martingales and then extend to local martingales.

If $M$ is $L^2$ bounded then it converges to $M_\infty \in L^2(\mathcal{F})$ and $M_t = \mathbb{E}[M_\infty|\mathcal{F}_t]$. The previous proposition gives

$$M_\infty = \mathbb{E}[M_\infty] + \int_0^\infty H_s dB_s$$

for a predictable process $H$ is $L^2(B)$. Hence we deduce that

$$M_t := \mathbb{E}[M_\infty] + \int_0^t H_s dB_s$$

Now assume $M$ is uniformly integrable, $M$ converges then to $M_\infty \in L^1(\mathcal{F})$. The space $L^2(\mathcal{F})$ is dense in $L^1(\mathcal{F})$ so there exists a sequence of $L^2$ bounded martingales $M^{(n)}$ such that $M_\infty^{(n)} \xrightarrow[n\to\infty]{L_1} M_\infty$. Doob's maximal inequality gives for any $\lambda > 0$

$$\mathbb{P}\left(\sup_t |M_t - M_t^{(n)}| > \lambda\right) \leq \frac{1}{\lambda}\mathbb{E}[|M_\infty - M_\infty^{(n)}|]$$

Using Borel-Cantelli we can extract a subsequence $M_{n_k}$ that converges uniformly to $M$ hence $M$ has a continuous version.

To extend to local martingales suffices to notice that it clearly has a continuous version (thanks to the previous point) and use the definiton : there exists a sequence of stopping times $T_k$ such that $M^{T_k}$ is bounded and use the first part of the proof. $\square$

The same result and proof are also valid in a multidimensional setting.

**Theorem 3.4.** *If $B$ is d-dimensional and $M$ is a local martingale with respect to the Brownian filtration then it has a continuous version and there exist predictable processes $H_i$ locally in $L^2(B_i)$ such that*

$$M_t = C + \sum_{j=1}^d \int_0^t H_j dB_s^{(j)}$$

A few interesting remarks should be made as this stage.

**Remark 3.5.**

(1) Notice that in Theorem 3.4 that $\langle M, B^j\rangle_t = \int_0^t H_j dt$ hence the processes $H_j$ are the Radon-Nikodym derivatives of $d\langle M, B^j\rangle_t$ with respect to the Lebesgue measure. In concrete examples these can be explicitly computed. For instance when $f$ is harmonic the representation of $f(B_t)$ is given by Itô's formula.

(2) The above result gives in particular a representation of variables $L^2(\mathcal{F})$ as stochastic integrals with $dB_s$, next we will see another kind of representation of this space.

We work again with the 1-dimensional case and we introduce some notation. Let $C_n$ be the $n$-dimensional polyhedral cone defined as

$$C_n := \{(s_1, \ldots, s_n) \in \mathbb{R}_+^n, s_1 > s_2 > \cdots > s_n\}$$

and $L^2(C_n)$ the space of real valued square-integrable functions on $C_n$ with respect to the Lebesgue measure. We denote by $\mathcal{S}_n$ the space spanned by function in the set $E_n$ which consists of $f$ on $C_n$ of the form $f(s) = f_1(s_1)\ldots f_n(s_n)$ where $f_i \in L^2(\mathbb{R}_+)$. This space if dense in $L^2(C_n)$ or in other words $E_n$ is a total family.

For $f = f_1 \ldots f_n \in E_n$ we define

$$J_n(f) = \int_{C_n} f(s)dB_s := \int_0^\infty f_1(s_1)dB_{s_1} \int_0^{s_1} f_2(s_2)dB_{s_2} \cdots \int_0^{s_{n-1}} f_n(s_n)dB_{s_n}$$

We have again the Itô's isometry (no so hard to check)

$$\|J_n(f)\|_{L^2(\mathcal{F})} = \|f\|_{L^2(C_n)}$$

Finally let $K_n$ be the smallest closed linear subspace of $L^2(\mathcal{F})$ that contains $J_n(f)$ for all $f \in E_n$ (or equivalently contains $J_n(\mathcal{S}_n)$). We call the space $K_n$ the $n$-th Wiener chaos space.

The map $J_n$ is defined on the set $E_n$ and it's not so hard to see that it is well defined on $\mathcal{S}_n$. Using the isometry property it can extended to $L^2(C_n)$ by taking limits since $\mathcal{S}_n$ is dense in $L^2(C_n)$ and the isometry property remains valid.

An interesting property of the spaces $K_n \subset L^2(\mathcal{F})$ is that for $n \neq m$ we have $K_n \perp K_m$ (not so hard to understand why: use Ito's isometry for usual stochastic integrals).

Now here is an interesting theorem

**Theorem 3.6.** *Let $K_0$ be the space of constants. Then one has*

$$L^2(\mathcal{F}) = \bigoplus_{n=0}^{\infty} K_n$$

*This means that for each $Y \in L^2(\mathcal{F})$ there exists a sequence $f^{(n)} \in L^2(C_n)$ such that*

$$Y = \mathbb{E}[Y] + \sum_{n=0}^{+\infty} J_n(f^{(n)})$$

*where the converges of the sum is in $L^2$.*

*Proof.* Let $f$ be a step function with compact support. We first show that the result is true for $\mathcal{E}_\infty^f = e^{\int_{\mathbb{R}_+} f(s)dB_s}$. We have

$$\mathcal{E}_\infty^f = 1+\sum_{n=1}^{\infty} \frac{1}{n!}\left(\int_{\mathbb{R}} f(s)dB_s\right)^n = 1+\sum_{n=1}^{\infty} \int_{\mathbb{R}_+} f(s_1)dB_{s_1} \int_0^{s_1} f(s_2)dB_{s_2} \cdots \int_0^{s_{n-1}} f(s_n)dB_{s_n}$$

Hence $\mathcal{E}_\infty^f = 1+\sum_{n=1}^{\infty} J_n(f^{(n)})$ where $f_n = f \ldots f$. Since $f$ is bounded with compact support the sum converges in $L^2$. So the property is also true for linear combinations of $\mathcal{E}^f_\infty$'s. Since the spaces these function span is dense in $L^2(\mathcal{F})$ and $L^2(\mathcal{F})$ is complete we deduce that the result is true for any variable $Y \in L^2(\mathcal{F})$ □

Notice that the first chaos space $K_1$ consists of Gaussian random variables. It is the closure of the spaces generated by the $B_t$'s. We have already encountered the decomposition into chaos spaces when we discussed the martingales $\exp(\theta B_t - \frac{1}{2}\theta^2 t)$. For $x$ and $t \geq 0$ we have

$$\exp\left(\theta x - \frac{1}{2}\theta^2 t\right) = \sum_{n=0}^{+\infty} H_n(x,t)\frac{\theta^n}{n!}$$

Then for an $L^2(\mathbb{R}_+)$ function $f$ we have

$$\exp\left(\theta \int_{\mathbb{R}_+} f(s)dB_s - \frac{1}{2}\theta^2 \int_{\mathbb{R}_+} f^2(s)ds\right) = \sum_{n=0}^{+\infty} H_n\left(\int_{\mathbb{R}_+} f(s)dB_s, \int_{\mathbb{R}_+} f^2(s)ds\right)\frac{\theta^n}{n!}$$

We actually have $\partial_x H(x,t) = H_{n-1}(x,t)$ thus

$$H_n\left(\int_0^t f(s)dB_s, \int_0^t f^2(s)ds\right) = \int_0^t H_{n-1}\left(\int_0^s f(s)dB_s, \int_0^s f^2(s)ds\right) f(s)dB_s$$

so clearly we have $H_n\left(\int_{\mathbb{R}_+} f(s)dB_s, \int_{\mathbb{R}_+} f^2(s)ds\right) \in K_n$. This actually another proof of the previous theorem. We refer to [PY18][Section 5] and references therein for more details.

It is known (we refer to [PY18][page 31]) that for $Y \in \bigoplus_{i=1}^n K_n$ there exists $\alpha > 0$ such that

$$\mathbb{E}[\exp(\alpha Y^{2/n})] < +\infty$$

Exercise 3.19 in chapter V of [RY13] provides a way to prove this.

In theorem 3.3 we have seen integral representation of local Brownian martingales. A natural question to ask is which martingale $M$ can be written as $M = H.B$ where $H$ is a predictable process and $B$ a Brownian motion for the filtration $(\mathcal{F}^M_t)$ generated by $M$. Here is a partial answer to this question in

**Proposition 3.7.** *If $M$ is a continuous local martingale such that the measure $d\langle M, M\rangle$ is almost surely equivalent to the Lebesgue measure, there exist an $(\mathcal{F}^M_t)$-predictable process $f_t$ which is $dt\otimes d\mathbb{P}$ strictly positive and an $(\mathcal{F}^M_t)$-Brownian motion $B$ such that*

$$d\langle M, M\rangle_t = f_t dt \text{ and } M_t = M_0 + \int_0^t f_s^{1/2} dB_s$$

*Proof.* Let $f$ the process defined as $f_t := \liminf_{n\to\infty} n(\langle M, M\rangle_t - \langle M, M\rangle_{t-1/n})$.
Since $d\langle M, M\rangle_t$ is a.s equivalent to the Lebesgue measure we deduce that $f$ is strictly positive a.s and that it's a predictable process. The process $f^{1/2}$ is locally in $L^2(M)$. Now let $B_t$ be the process defined as

$$B_t := \int_0^t f_s^{-1/2} dM_s$$

$B$ is then a continuous local martingale with $\langle B, B\rangle_t = t$ thus $B$ is a Brownian motion. □

The condition $d\langle M, M\rangle_t$ is equivalent to the Lebesgue measure is crucial. It it not enough that it is absolutely continuous with respect to $dt$ because the filtration of $M$ might not be big enough to support a Brownian motion (just think of a constant martingale). However one can avoid this problem if we have a Brownian motion $B'$ independent of $M$ and set

$$B_t := \int_0^t 1_{f_s>0} f_s^{-1/2} dM_s + \int_0^t 1_{f_s=0} dB'_s$$

Using Lévy's characterization theorem, $B$ is a Brownian motion and one has $M_t = \int_0^t f_s^{1/2} dB_s$. In other words the theorem still holds if we enlarge the probability space to provide ourselves with an independent Brownian motion. This result can be proved in the multidimensional case as the following theorem states.

**Theorem 3.8.** *Let $M = (M_1, \dots, M_d)$ be a continuous local martingale such that $d\langle M^{(i)}, M^{(i)}\rangle_t \ll dt$ for every $i$. Then with a possible enlargement of probability space, there exists a $d$-dimensional Brownian motion $B$ and a $d \times d$ matrix valued process $\alpha$ in $L^2_{loc}(B)$ such that*

$$M_t = M_0 + \int_0^t \alpha_s dB_s$$

*Sketch of the proof.* We reduce to the case $M_0 = 0$ and use the previous theorem and linear algebra. We refer to [RY13][Page 203] for details of the proof. □

## 4. Integral representation

We start this section with a definition.

**Definition 4.1.** A continuous local martingale $X$ has the predictable representation property (denoted PRP) if for any $(\mathcal{F}_t^X)$-local martingale $M$ there exists an $(\mathcal{F}_t^X)$-predictable process $H$ such that

$$M_t = M_0 + \int_0^t H_s dX_s$$

In section 3 we have seen that Brownian local martingales can be written as stochastic integrals of predictable processes. Hence the Brownian motion has the PRP. This is actually a property of the Wiener measure on the space $C([0, +\infty), \mathbb{R})$ of real valued continuous functions on $[0, +\infty)$. Before we set things up here is a useful decomposition result.

**Lemma 4.2.** *If $X$ is any continuous local martingale, then for every $(\mathcal{F}_t^X)$-continuous local martingale $M$ vanishing at $0$ there exists a unique $(\mathcal{F}_t^X)$-predictable process $H$ such that the process $L = M - H.X$ satisfies $\langle L, X\rangle = 0$. This means $M$ can be decomposed as*

$$M = H.X + L \text{ with } \langle X, L\rangle$$

*Proof.* The uniqueness is not so hard to see because this decomposition is linear in $H$. To prove existence we consider a sequence $T$ increasing to $+\infty$ of stopping times that reduces both $M$ and $X$ to $L^2$-bounded martingales. We denote by $H_0^2$ the Hilbert space of continuous $L^2$-bounded $(\mathcal{F}_t^X)$-martingales vanishing at 0. The space $G := \{H.X^T, H \in L^2(X^T) \text{ predictable } \}$ is closed inside $H_0^2$. So we can apply the projection theorem on a closed subspace of a Hilbert space to obtain

$$M^T = \overline{H}.X^T + \overline{L}$$

where $\overline{L} \in G^{\perp}$. For any bounded stopping time $S$ we have

$$\mathbb{E}[X_S^T \overline{L}_S] = \mathbb{E}[X_S^T \mathbb{E}[\overline{L}_\infty] \mathcal{F}_S] = \mathbb{E}[X_S^T \overline{L}_\infty] = 0$$

The last equality is due to the fact that $X_S^T = X_\infty^{T\wedge S}$ and $X^{T\wedge S} \in G$. Hence the process $X^T\overline{L}$ is a martingale and $\langle X^T, \overline{L}\rangle = \langle X, \overline{L}\rangle^T = 0$. Thanks to the uniqueness result we can extend the processes $\overline{H}, \overline{L}$ to processes $H, L$ that satisfy the desired property. □

Now we set things up in such ta way that we discuss probability measure with representation properties instead of processes. We work on the Wiener space $\mathbf{W} := C(\mathbb{R}_+, \mathbb{R})$ on which the coordinate process is denoted $X$ and we put $\mathcal{F}_t^0 = \sigma(X_s, s \leq t)$. Let $\mathcal{H}$ be the set of probability measures of $\mathbf{W}$ such that $X$ is a local martingale. If $\mathbb{P} \in \mathcal{H}$ we call $\mathcal{F}_t^{\mathbb{P}}$ the smallest right-continuous filtration of $\mathbb{P}$ that is complete for $\mathbb{P}$ and such that $\mathcal{F}_t^0 \subset \mathcal{F}_t^{\mathbb{P}}$.

The PRP now is formulated as a property of the measure $\mathbb{P}$ as follows: any $\mathcal{F}^{\mathbb{P}}$-local martingale may be written as $M = H.X$ where $H$ is $\mathcal{F}^{\mathbb{P}}$-predictable and the stochastic integration is taken with respect to $\mathbb{P}$.

We denote by $\mathcal{K}$ the subset of $\mathcal{H}$ consisting of probability measures for which $X$ is a martingale. It is not very hard to see that sets $\mathcal{K}$ and $\mathcal{H}$ are convex.

**Definition 4.3.** A probability measure $\mathbb{P}$ of $\mathcal{K}$ (resp $\mathcal{H}$) is called extremal if whenever $\mathbb{P} = \alpha\mathbb{P}_1 + (1-\alpha)\mathbb{P}_2$ with $\alpha \in (0,1)$ and $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{K}$ (resp. $\mathcal{H}$) then $\mathbb{P}_1 = \mathbb{P}_2$.

The following discussion relates the PRP to extremality. Let's start with a useful measure theory result

**Theorem 4.4** (Douglas). *Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathcal{L}$ a set of real valued $\mathcal{F}$-measurable functions and $\mathcal{L}^*$ the vector space generated by 1 and $\mathcal{L}$.*

*If $\mathcal{K}_{\mathcal{L}}$ is the set of probability measures $\mu$ on $(\Omega, \mathcal{F})$ such that $\mathcal{L} \subset L^1(\mu)$ and $\int f d\mu = 0$ for all $f \in \mathcal{L}$ then $K_{\mathcal{L}}$ is convex and $\mu$ is extremal in $K_{\mathcal{L}}$ if and only if $\mathcal{L}^*$ is dense in $L^1(\mu)$.*

*Proof.* The proof is not very hard, we refer to [RY13][page 210]. □

As an application of this theorem we consider the space $(\mathbf{W}, \mathcal{F}_\infty^0)$ and the set $\mathcal{L}$ of random variables $1_A(X_t - X_s)$ where $0 \leq s < t$ and $A$ in $\mathcal{F}_s^0$. The set $\mathcal{K}_{\mathcal{L}}$ of the previous theorem is in this case exactly the set $\mathcal{K}$ of probabilities for which $X$ is a martingale. We shall use this measure theory result to show the

**Proposition 4.5.** *If $\mathbb{P}$ is extremal in $\mathcal{K}$ then any $(\mathcal{F}_t^{\mathbb{P}})$-local martingale has a continuous version*

*Proof.* Notice that it is enough to show that for any $Y \in L^1(\mathbb{P})$, the cadlag martingale $\mathbb{E}_{\mathbb{P}}[Y|\mathcal{F}_t^{\mathbb{P}}]$ has a continuous version. This is thanks to the 3 step argument that we have already encountered ( passing from uniformly integrable martingales to local martingales ).

For $Y \in \mathcal{L}^*$ it is not hard to see that the previous property is true when $Y \in \mathcal{L}$ (hence also in $\mathcal{L}^*$). The previous measure theory result implies that $\mathcal{L}^*$ is dense in $L^1(\mathbb{P})$. Hence for a variable $Y \in L^1(\mathbb{P})$ there is an approximating sequence $Y_n \in \mathcal{L}^*$. Using the maximal inequality, for any $\epsilon > 0$

$$\mathbb{P}\left[\sup_{s\leq t}\left|\mathbb{E}_{\mathbb{P}}[Y_n|\mathcal{F}_s^{\mathbb{P}}] - \mathbb{E}_{\mathbb{P}}[Y|\mathcal{F}_s^{\mathbb{P}}]\right| \geq \epsilon\right] \leq \frac{1}{\epsilon}\mathbb{E}[|Y_n - Y|]$$

Borel-Cantelli allows to extract a uniformly convergent subsequence from the $Y_n$'s which finishes the proof.

□

**Theorem 4.6.** *The probability measure $\mathbb{P}$ is extremal in $\mathcal{K}$ if and only if $\mathbb{P}$ has the PRP and $\mathcal{F}_0^{\mathbb{P}}$ is $\mathbb{P}$-trivial.*

*Proof.* If $\mathbb{P}$ is extremal then $\mathcal{F}_0^{\mathbb{P}}$ (otherwise one can decompose $\mathbb{P}$ into a nontrivial convex combination of measures in $\mathcal{K}$). Suppose that the PRP does not hold for $\mathbb{P}$. By lemma 4.2 and the previous proposition there exists a continuous $(\mathcal{F}_t^{\mathbb{P}})$-local martingale $L$ such that $\langle X, L\rangle = 0$. By stopping, since we have $\langle X, L^T\rangle = \langle X, L\rangle^T$ for any stopping time, we may assume that $L$ is bounded by some constant $k$. Now let $\mathbb{P}_1 = (1+L_\infty/(k+1))\mathbb{P}$ and $\mathbb{P}_1 = (1-L_\infty/(k+1))\mathbb{P}$. We then have $\mathbb{P} = (\mathbb{P}_1+\mathbb{P}_2)/2$ is a non trivial convex combination and also $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{K}$ since $\mathbb{P} \in \mathcal{K}$ and $L$ is a bounded martingale. This contradicts the extremality of $\mathbb{P}$.

Conversely, assume that $\mathbb{P}$ has the PRP and $\mathcal{F}_0^{\mathbb{P}}$ is $\mathbb{P}$-a.s trivial, and that $\mathbb{P} = \alpha\mathbb{P}_1 + (1-\alpha)\mathbb{P}_2$ with $\alpha \in (0,1)$ and $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{K}$. The derivative $\frac{d\mathbb{P}_1}{d\mathbb{P}}|_{\mathcal{F}_t}$ is a $\mathbb{P}$-martingale has a continuous version $L$ since $\mathbb{P}$ has the PRP. $XL$ is also a continuous martingale hence $\langle X, L\rangle = 0$. Since $\mathbb{P}$ has the PRP we can write $L_t = L_0 + \int_0^t H_s dX_s$ and one has $\int_0^t H_s d\langle X, X\rangle_s$. Then $\mathbb{P}$-a.s we have $H_s = 0$ $d\langle X, X\rangle$.a.e. Then $L_t = L_0$ and $L$ is constant $\mathbb{P}$-a.s. Since the starting sigma-algebra $\mathcal{F}_0^{\mathbb{P}}$ is trivial one has $L = 1$ a.s. Hence $\mathbb{P} = \mathbb{P}_1$ and thus $\mathbb{P}$ is extremal. □

This result can be extended to the set $\mathcal{H}$ of measures that make $X$ a local martingale. The proof is rather technical so we will not discuss it here and we refer the reader to [RY13][page 211] for a fairly detailed argument.

**Theorem 4.7.** *The probability measure $\mathbb{P}$ is extremal in $\mathcal{H}$ if and only if $\mathbb{P}$ has the PRP and $\mathcal{F}_0^{\mathbb{P}}$ is $\mathbb{P}$-trivial*

*Proof.* Omitted □

This result implies that the Wiener measure on $\mathbf{W}$ is extremal. One can actually prove this fact very easily using Lévy's characterization of Brownian motion.

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