A (very) brief introduction to symmetric spaces

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Abstract. This text consists of reading notes that aim to introduce the theory of symmetric spaces and their relation to Lie groups. The goal is to give intuition and focus on the geometric aspect as much as possible .

1. Introduction and notation

There are many approaches to the concept of geometry. Perhaps one of the most insightful ones is the approach suggested by Felix Klein: considering a space $\mathcal{M}$ with a group of transformations G. This conception provides striking links between algebra and geometry.

The theory symmetric spaces falls into this paradigm and is a meeting points of many branches in mathematics namely differential geometry [Hel01], number theory [JLR93] , harmonic analysis [Ion00, vdBS99] and statistics [SM21] to name a few.

The literature on the topic is very vast and these few pages will certainly not give it justice. So our focus will be to present the main ideas and exhibit some examples where one might have encountered symmetric spaces without knowing so. Our hope is that these notes will grow with time with a view towards the non-archimedean side of the theory (Bruhat-Tits buildings and Berkhovic spaces) which is more prominent in arithmetic geometry and number theory. But for the present notes, we stick to the setting of Riemannian geometry.

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Our main object of interest throughout this text is a Riemannian C^∞ manifold $(\mathcal{M}, q)$ together with its metric q. We denote by $\mathrm{I}(M)$ its isometry group. We endow this group with the *compact-open topology.* If C, U are respectively compact and open in $\mathcal{M}$ and

$$W(C,U) \coloneqq \{g \in \mathrm{I}(\mathcal{M})\colon g.C \subset U\},$$

the compact-open topology is the smallest topology for which all such sets $W(C,U)$ are open. It is not so hard to check that, endowed with this topology, the group $\mathrm{I}(\mathcal{M})$ is a Hausdorff space. The connected component containing the identity will be denoted by $\mathrm{I}_0(\mathcal{M})$. Moreover, we have the following (non-trivial) theorem:

THEOREM 1.1. — *The group* $\mathrm{I}(\mathcal{M})$*, endowed with the compact-open topology, is a locally compact topological group.*

The notes are organized as follows. In section 2 we present the first definitions and preliminaries. Section 3 is dedicated to the description of symmetric spaces as quotients of Lie groups. In section 4, we go through a couple of examples to illustrate the theory. In section 5, we talk about Gelfand pairs (or symmetric pairs) and shift gears to geodesics, curvature and type of a symmetric space.

2. Preliminaries

Throughout this text, we assume a basic level of familiarity with concepts from differential and mainly Riemannian geometry. Of the many sources in the literature, we refer the reader for example to [Hel01, Zil10].

DEFINITION 2.1. — *Let* $(\mathcal{M}, q)$ *be a Riemannian manifold.*

(1) The manifold $\mathcal{M}$ *is called symmetric if for all* $p \in \mathcal{M}$ *there exists an isometry* $s_p\colon \mathcal{M} \to \mathcal{M}$ *such that* $s_p(p) = p$ *and* $d_p(s_p) = -\mathrm{Id}$ *on the tangent space* $T_p\mathcal{M}$*. In this case we also say that* $\mathcal{M}$ *is globally symmetric.*

(2) We say that $\mathcal{M}$ *is locally symmetric if for any point* $p \in \mathcal{M}$ *there exists* $r > 0$ *and an isometry* $s_p\colon B(p,r) \to B(p,r)$ *of the ball* $B(r,p)$ *of radius* r *around* p *such that* $s_p(p) = p$ *and* $d_p(s_p) = -\mathrm{Id}$ *on* $T_p\mathcal{M}$*.*

Let us now examine some immediate properties of symmetric spaces.

PROPOSITION 2.2. — *Let $(\mathcal{M}, g)$ be a symmetric space. Then the following properties hold*

(a) *If $\gamma\colon (-a, a) \to \mathcal{M}$ is a geodesic and $\gamma(0) = p$ then we have $s_p(\gamma(t)) = \gamma(-t)$ i.e. s_p reverses the direction of geodesics passing through $p \in \mathcal{M}$.*

(b) *$\mathcal{M}$ is complete i.e. geodesics are defined for all $t \in \mathbb{R}$ (equivalently, also as a metric space).*

(c) *$\mathcal{M}$ is a homogeneous space i.e. for any $p, q \in \mathcal{M}$ there exists an isometry $f\colon \mathcal{M} \to \mathcal{M}$ such that $f(p) = q$.*

Proof. — We refer the reader to [Zil10] for a proof. □

LEMMA 2.1. — *Let $(\mathcal{M}, q)$ be a Riemannian manifold. Then if $\mathcal{M}$ is homogeneous and there exists a symmetry in a point p then the space $\mathcal{M}$ is symmetric.*

Proof. — Let $p \in \mathcal{M}$ with involution s_p and $q \neq p \in \mathcal{M}$ another point. Since $\mathcal{M}$ is homogeneous, there exists an isometry $\varphi \in \mathrm{I}(\mathcal{M})$ such that $\varphi(p) = q$. Then the isometry $s_q \coloneqq \varphi \circ s_p \circ \varphi^{-1}$ is an involution as q (check that $d_q(s_q) = -\operatorname{Id}$). n □

Remark 2.3. — A basic fact on symmetric spaces is that if $(\mathcal{M}, p)$ is a symmetric space and $p \in \mathcal{M}$ a point then the symmetry s_p is unique. Also notice that $s_p^2 = \operatorname{Id}_{\mathcal{M}}$.

To get a taste of what symmetric spaces are, let us examine a couple of simple examples.

EXAMPLE 2.4. — (1) *Let $n \geqslant 2$ be an integer and let $S^{n-1} \subset \mathbb{R}^n$ be the unit sphere of dimension $n-1$. We can easily make S^{n-1} into a Riemannian manifold using the usual Riemannian metric q. For each point $x \in S^{n-1}$ we define the symmetry*

$$s_x(y) \coloneqq 2\langle x, y\rangle x - y.$$

Where $\langle \cdot, \cdot \rangle$ is the usual scalar product on $\mathbb{R}^n$. The reader can check that inded s_x is a smooth isometry of the sphere, that $s_x(x) = x$ and that $d_x(s_x) = -\operatorname{Id}$.

(2) *Let G be a Lie group and let q a bi-invariant metric on G. Then the Riemannian manifold (G, q) is a symmetric space. To see why first notice that the map $s_e\colon g \mapsto g^{-1}$ is a symmetry around the identity this is because naturally*

$$s_e(e) = e \quad \text{and } d_e(s_e) = -\operatorname{Id}.$$

The fact that s_e is an isometry stems from q being bi-invariant. For a general point $g \in G$ it is not hard to see that the map $s_g \colon h \to (gh)^{-1} = h^{-1}g^{-1}$.

Let us now introduce one-parameter families of isometries which will be of key importance later on: *transvections*.

DEFINITION 2.5 (transvections). — *Let (M, q) be a symmetric space and $\gamma \colon \mathbb{R} \to \mathcal{M}$. The transvection T_t^γ is the isometry defined as follows*

$$T_t^\gamma := s_{\gamma(t/2)} \circ s_{\gamma(0)}.$$

PROPOSITION 2.6. — *Let $(\mathcal{M}, q)$ be a symmetric space and $\gamma \colon \mathbb{R} \to \mathcal{M}$ be a geodesic. Then the following hold:*

(i) $T_t^\gamma(\gamma(s)) = \gamma(t+s)$ *i.e.* T_t^γ *acts as a translation along the geodesic* γ.
(ii) $(T_t^\gamma)_{t\in\mathbb{R}}$ *is a one parameter subgroup of* $\mathrm{I}(\mathcal{M})$.

Proof. — The proof is not so hard, we refer the reader to [Zil10] for a detailed proof. □

Remark 2.7. — Part (*ii*) is the previous proposition means that geodesics on $\mathcal{M}$ are images of one-parametrer groups of isometries of $\mathcal{M}$. Also, notice that if $p \neq q \in \mathcal{M}$, then there exists a geodesic γ such that $\gamma(0) = p$ and $\gamma(1) = q$. So we have

$$T_1^\gamma(p) = T_1^\gamma(\gamma(0)) = \gamma(1) = q.$$

So, since $T_1^\gamma \in \mathrm{I}_0(\mathcal{M})$ we deduce that $\mathrm{I}_0(\mathcal{M})$ acts transitively on $\mathcal{M}$.

EXAMPLE 2.8. — *An important example to keep in mind is that of the hyperbolic plane which we choose to model by the Poincare upper-half plane $\mathbb{H} = \{z \in \mathbb{C} \colon \mathrm{Im}(z) > 0\}$ with arc-length $d\alpha(z) = \frac{1}{\mathrm{Im}(z)}dz$. Geodesics in $\mathbb{H}$ are circle arcs. For the point i the involution s_i is given by $s_i \colon z \to \frac{-1}{z}$. For a general $z \in \mathbb{H}$, we get the involution s_z by conjugating the involution s_i with the isometry (it is a Möbeius transformation) $w \to \frac{w - \mathrm{Re}(z)}{\mathrm{Im}(z)}$.*

Transvections, as 1-parameter groups of isometries, are open half circles (since geodesics are). As we shall see in section 3, we can also describe $\mathbb{H}$ as a quotient of a Lie group.

3. Group quotients and Cartan involutions

In this section we discuss symmetric spaces as quotients of their isometry group by a compact subgroup fixing a point. This shall allow us to introduce the notion of Cartan involutions.

Throughout this section we consider a symmetric space $(\mathcal{M}, q)$ and we fix a point $p \in \mathcal{M}$. We denote by $G = I_0(\mathcal{M})$ and K the stabilizer of the point p. We can then see that there is a one to one map between $\mathcal{M}$ and the quotient G/K. We start with the following fundamental theorem:

THEOREM 3.1. — *The following hold*

(i) *The group K is compact,*
(ii) *The quotient G/K is analytically diffeomorphic to $\mathcal{M}$ under the map $gK \mapsto g \cdot p_0$ where $gK \in G/K$,*
(iii) *The mapping $\sigma\colon G \to G, g \mapsto s_p \circ g \circ s_p$ is an involutive automorphism of G. Moreover, if G^σ is the closed group of fixed points of σ and $G^\sigma_\circ$ its connected components then $G^\sigma_\circ \subset K \subset G^\sigma$.*

Proof. —

For the sake of brevity, we omit the proof of (i) and (ii). We refer the interested reader to [Hel01]. For (iii), the fact that σ is an involution stems from the fact that $s_p^2 = \mathrm{Id}_{\mathcal{M}}$. Let $h \in K$, then $\sigma(h)(p) = s_p \circ h \circ s_p(p) = h(p) = p$. So we deduce that $K \subset G^\sigma$. Let (g_t) be a 1-parameter subgroup of $G^\sigma_\circ$. We have $\sigma(g_t) = g_t$, so $s_p \circ g_t \circ s_p = g_t$. Applying this to p yields $s_p(g_t(p)) = g_t(p)$ for all t. But p is isolated as a fixed point of s_p and $g_t(p)$ tends to p as $t \to 0$. So $g_t(p) = p$ for all t. Hence $g_t \in K$. Hence $G^\sigma_\circ \subset K$.

□

DEFINITION 3.1. — *The involution σ is called a Cartan involution of the symmetric space $\mathcal{M} \cong G/K$.*

Let $d\sigma$ be the differential of the involution σ at $e \in G$. Since σ is an involution the differential $d\sigma$ is also an involution on the tangent space $\mathfrak{g} = T_eG$. Let $\mathfrak{k}, \mathfrak{p}$ the $+1$ and -1 eigenspaces of $d\sigma$ respectively. We have the following

PROPOSITION 3.2. — *The space $\mathfrak{k}$ is the Lie algebra of K and we have*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

Furthermore , $\mathrm{Ad}_K(\mathfrak{p}) \subset \mathfrak{p}$, *in particular* $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$.

Proof. — Since $[\mathfrak{k},\mathfrak{k}]$ then $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$ and since $G^\sigma_\circ \subset K \subset G^\sigma$ we deduce that K and G^σ have the same Lie algebra which is the $+1$ eigenspace of $d\sigma$ i.e. it is $\mathfrak{k}$. Sine $d\sigma$ is a Lie algebra homomorphism we deduce that $[\mathfrak{p},\mathfrak{p}] = \mathfrak{k}$.. Notice that we have $d\sigma \circ \mathrm{Ad}(g) = \mathrm{Ad}(\sigma(g)) \circ d\sigma$. So if $h \in K$ and $x \in \mathfrak{p}$ i.e. $\sigma(h) = h$ and $d\sigma(x) = -x$, this means that $d\sigma(\mathrm{Ad}(h)(x)) = \mathrm{Ad}(\sigma(h))(d\sigma(x)) = -\mathrm{Ad}(\sigma(h))(x) = -\mathrm{Ad}(h)(x)$. So $\mathrm{Ad}(h)(x) \in \mathfrak{p}$ by definition of $\mathfrak{p}$. □

Such a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is called a ***Cartan decomposition***. We now give a converse of theorem 3.1.

Proposition 3.3. — *Let G be a connected Lie group and $\sigma\colon G \to G$ an involutive automorhism of G such that $G^\sigma_\circ$ is compact. Then for any compact subgroup K with $G^\sigma_\circ \subset K \subset G^\sigma$, the homogeneous space G/K, equipped with any G-invariant metric, is a symmetric space, and such metrics exist.*

Proof. — Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition from above. Homogeneous metrics on the quotient G/K are given by Ad_K-invariant inner products on $\mathfrak{p}$. Since K is compact, such inner product do indeed exist. Next, we show that any such metric is actually symmetric. Since G/K is homogeneous, we only need to find an involution in one point. It it not so hard to see that the induced map $\overline{\sigma}\colon G/K \to G/K, gK \to \sigma(g)K$ is an involution fixing the point $eK \in G/K$.

□

Proposition 3.4. — *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a decomposition (as vector spaces) satisfying*

$$[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p} \quad \textit{and } [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k},.$$

Then, if G is a simple connected Lie group such that $Lie(G) = \mathfrak{g}$ and $K \subset G$ the connected subgroup of G with Lie algebra $\mathfrak{k}$, we have:

(1) There exists an involution $\sigma\colon G \to G$ such that $K = G^\sigma_\circ$,

(2) If K is compact, then every G-invariant metric on G/K is a symmetric space..

Proof. — For the sake of brevity the proof (which is not too hard: linear algebra) is omitted.

□

Proposition 3.5. — *The following hold:*

(1) If $\mathcal{M}$ is a symmetric space and $\mathcal{N} \subset \mathcal{M}$ is a submanifold such that for any $p \in \mathcal{M}$ we have $s_p(\mathcal{N}) = \mathcal{N}$ then $\mathcal{N}$ is totally geodesic and symmetric.
(2) Let $\sigma \colon G \to G$ be an involutive automorphism and $\mathcal{M} = G/K$ the corresponding symmetric space. If $H \subset G$ with $\sigma(H) \subset H$, then $H/(H \cap K)$ is a symmetric space such that $H/(H \cap K)$ is totally geodesic in G/K.

Proof. —

(1) By definition $\mathcal{N} \subset \mathcal{M}$ is totally geodesic if every geodesic in $\mathcal{N}$ is also a geodesic in $\mathcal{M}$. This follows because the involution s_p preserves the tangent space $T_p\mathcal{N}$ inside $T_p\mathcal{M}$ for all $p \in \mathcal{N}$.
(2) This follows immediately from (1).

□

4. Some fundamental examples

TIn this section we go through some fundamental example that one should keep in mind to gain intuition. We try to focus on the geometric aspect of things and keep technical details under the rug (we give references whenever necessary).

4.1. The positive definite cone

Let $n \geqslant 2$ be an integer and $\mathcal{M}_n$ the cone of positive definite matrices i.e.

$$\mathcal{M}_n := \{A \in \mathbb{R}^{n\times n} \colon \langle x, Ax\rangle > 0, \quad \text{for all non-zero} \;\; x \in \mathbb{R}^n\}.$$

The spae $\mathcal{M}_n$ is an open set in $\mathbb{R}^{n\times n}$ and is thus a Riemannian manifold equipped with the metric $\langle \cdot, \cdot \rangle_A$ defined by

$$\langle X, Y\rangle_A := \mathrm{Tr}(A^{-1}XA^{-1}Y).$$

The Lie group $\mathrm{GL}_n(\mathbb{R})$ acts on $\mathcal{M}_n$ transitively by $g.A = gAg^T$ and it is not very hard to see that this action is an isometry of $\mathcal{M}_n$. The stabilizer of the identity $\mathrm{Id}_n \in \mathcal{M}_n$ is the orthogonal group $O_n(\mathbb{R})$ and hence $\mathcal{M}_n = \mathrm{GL}_n(\mathbb{R})/O_n(\mathbb{R})$. In terms of connected Lie groups one has

$$\mathcal{M}_n := \mathrm{GL}_n^+(\mathbb{R})/SO_n(\mathbb{R}).$$

In the cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, the space $\mathfrak{k}$ is the space of skew symmetric matrices (Lie algebra of $SO_n(\mathbb{R})$) and $\mathfrak{p}$ is the space of symmetric matrices.

4.2. Symplectic geometry: Lagrangian subspaces of $\mathbb{R}^{2n}$

Let V be a vector space over $\mathbb{R}$ of dimension $2n$ and ω a symplectic form on V. We call a subspace L of V Lagrangian if $\dim(L) = n$ and $\omega_{|L} = 0$. There exists a symmplectic basis (or Darboux basis) $p_1, \ldots, p_n, q_1, \ldots, q_n$ such that $\omega = \sum_{i=1}^{n} p_i \wedge q_i$. Thus there exists a scalar product $\langle \cdot, \cdot \rangle$ together with a complex structure $J \in \mathrm{End}(V)$ (i.e. $J^2 = -\mathrm{Id}$) such that $\omega(x, y) = \langle x, Jy \rangle$. So L is a lagrangian space if and ony if $L \perp JL$.

The set $\mathcal{M}$ of Lagrangian spaces in V can be equipped with the structure of a symmetric space as follows. To simplify the matter we identify (V, ω, J) with $(\mathbb{R}^{2n}, \omega_0, J_0)$ via a Darboux basis. The symplectic group $\mathrm{Sp}_n(\mathbb{R})$ (leaving the form ω invariant) takes Lagrangian spaces to Lagrangian spaces. We can see without much difficulty that the group $\mathrm{U}_n(\mathbb{C})$ is a subgroup of $\mathrm{Sp}_n(\mathbb{R})$ (actually $\mathrm{U}_n(\mathbb{C}) = \mathrm{Sp}_n(\mathbb{R}) \cap O_{2n}(\mathbb{R})$). Moreover, $\mathrm{U}_n(\mathbb{C})$ acts transitively on $\mathcal{M}$ and, if $L_0 = \mathbb{R}^n$ is the space spanned by the first n vectors in the Darboux basis then the stabilizer of L_0 in $\mathrm{U}_n(\mathbb{C})$ is the group $O_n(\mathbb{R})$ (embedded diagonally in $\mathrm{Sp}_n(\mathbb{R})$). We then deduce that $\mathcal{M} = \mathrm{Sp}_n(\mathbb{R})/O_n(\mathbb{R})$. Or in terms of connected groups

$$\mathcal{M}^\circ = \mathrm{U}_n(\mathbb{C})/\mathrm{SO}_n(\mathbb{R}).$$

There is not shortage of example and situations where symmetric spaces arise and are relevant. We refer the reader to [Hel01] for more examples.

5. Symmetric pairs

After having described symmetric spaces as quotients of groups, it is relevant to give the following definition

DEFINITION 5.1. — *A symmetric pair (G, K) is a pair consisting of a Lie group G, a compact subgroup K together with an involution σ of G suh that $G^\sigma_\circ \subset K \subset G^\sigma$ and G acts almost effectively on G/K.*

I did not finish this part due to lack of time. I am also interested in studying non-archimeadean symmetric spaces and p-adic Lie groups which I hope to add to these notes as they grow.

Bibliography

[Hel01] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 34 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Corrected reprint of the 1978 original.

[Ion00] Alexandru D. Ionescu. Fourier integral operators on noncompact symmetric spaces of real rank one. *J. Funct. Anal.*, 174(2):274–300, 2000.

[JLR93] Hervé Jacquet, King F. Lai, and Stephen Rallis. A trace formula for symmetric spaces. *Duke Math. J.*, 70(2):305–372, 1993.

[SM21] Salem Said and Jonathan H. Manton. Riemannian barycentres of Gibbs distributions: new results on concentration and convexity in compact symmetric spaces. *Inf. Geom.*, 4(2):329–362, 2021.

[vdBS99] Erik P. van den Ban and Henrik Schlichtkrull. Fourier inversion on a reductive symmetric space. *Acta Math.*, 182(1):25–85, 1999.

[Zil10] Wolfgang Ziller. Lie groups. representation theory and symmetric spaces, 2010. available at https://www2.math.upenn.edu/~wziller/math650/LieGroupsReps.pdf.